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# ON THE PROBLEM OF CONSTRUCTION OF THE UNDERGROUND CONTOUR OF WATER-DEVELOPMENT WORKS WITH PORTIONS OF CONSTANT FLOW VELOCITY

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*The underground contour of a sunk rectangular apron whose angles are rounded off by the curves of constant filtration rate is constructed in the case where the water-permeable base is underlain by a confining layer with a curvilinear roof, characterized by a constant flow velocity, too. The corresponding boundary-value problem is solved by a semiinverse application of the velocity-hodograph method. The cases of the apron with a horizontal insert in flow and the rabbet in flow are studied in detail. The results of numerical calculations are given; the influence of the physical parameters of the model on the shape and dimensions of the underground contour of the dam is analyzed.*

**Introduction.** The issues of necessity and advisability of using smooth underground contours in water development were first the focus of [1–3]. These ideas received a large development effort in [4, 5] where the inverse approach was first used. This made it possible to construct the underground contour of a curvilinear apron characterized by the constancy of flow velocity in the case where a water-permeable base is underlain by a confining layer with a horizontal roof and to consider a number of mixed boundary-value problems where some portions of the underground contour are considered to be assigned in shape, and others are determined from the condition of constancy of the filtration rate. This resulted in efficient solutions for the cases of a rectangular apron whose angles are rounded-off by the curves of constant flow velocity and a rabbet rounded off at its lower part.

Works [4, 5] gave rise to the entire line of inquiry — determination of the contour of water-development works from their assigned properties — and produced numerous investigations of flows of this kind that mainly belong to the Kazan' school of mechanics (see, e.g., [6–9]).

Unlike [4, 5], below, not only do we consider the construction of the smooth underground contour of a sunk rectangular apron but we also determine the outline of the underlying water-permeable base characterized by a constant filtration rate, too. The corresponding multiparametric mixed problem of the theory of analytical functions is solved with a semiinverse application of the first variant of a velocity hodograph [10–12]. We study in detail the limiting cases of flow due to the degeneration of the conformal-mapping parameters contained in the solution: the diagrams of the apron and rabbet (or cutoff apron) in flow.

**Formulation of the Problem.** We consider plane steady-state flow under the water-permeable underground contour of a sunk apron $ABCC_1B_1A_1$ (Fig. 1). Let the contour of the apron base $AA_1$ consist of two vertical segments AB and $A_1B_1$ of equal length, the central horizontal segment $CC_1$, and the adjacent arcs of curves BC and $B_1C_1$ with a constant flow velocity $|w| = v_0$. The flow region is bounded from below by a curvilinear confining layer $FF_1$ on which the filtration rate is constant, too $|w| = u_0$ $(0 < u_0 < v_0)$. It is assumed that the boundaries of the upper and lower pools are horizontal, the ground is homogeneous, and the motion obeys Darcy's law with a known filtration coefficient $\kappa$ = const. The head $H$ acting on the works, the flow velocity $v_0$, the filtration flow rate $Q$, and the apron widh $2l$ and sinking (occurrence depth) $d$ are considered to be assigned.

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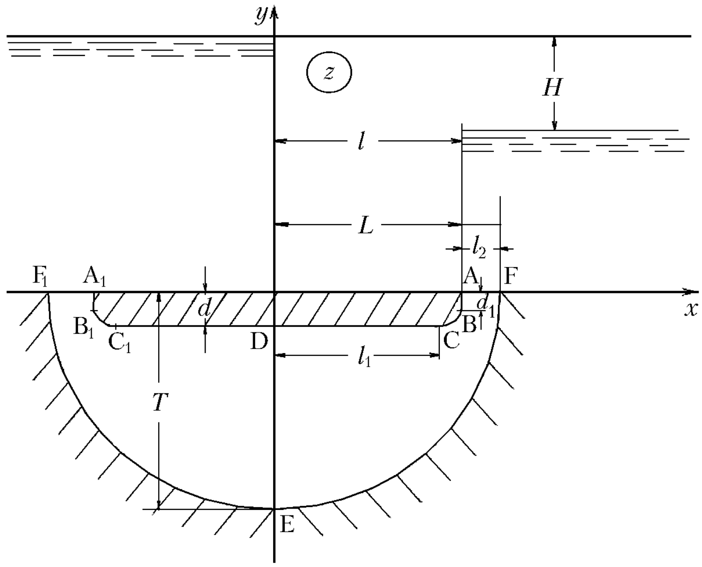


Fig. 1. Diagram of the underground contour of water-development works with portions of constant flow velocity.

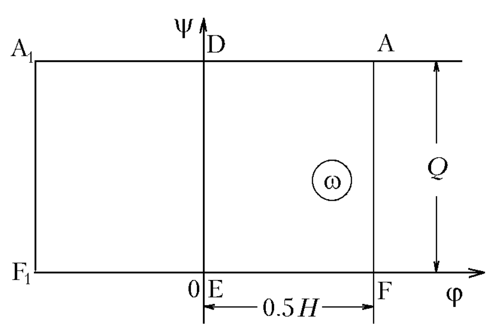


Fig. 2. Region of the complex flow potential.

We introduce a complex motion potential $\omega = \varphi + i\psi$ (the range of variation in the variable $\omega$ is presented in Fig. 2) and a complex coordinate referred to $kH$ and $H$ respectively. The problem is in determining the position of curves BC, $B_1C_1$, and $FF_1$ with the following boundary conditions:

$$A_1F_1: y = 0, \quad \varphi = -0.5H; \quad A_1B_1: x = -l, \quad \psi = Q,$$

$$C_1C: y = -d, \quad \psi = Q; \quad AB: x = l, \quad \psi = Q, \tag{1}$$

$$AF: y = 0, \quad \psi = 0.5H; \quad FF_1: \psi = 0,$$

so that the filtration rate along the curvilinear portions of the apron's underground contour BC and $B_1C_1$ and the confining layer $FF_1$ has constant values $v_0$ (assigned) and $u_0$ (sought) respectively.

**Construction of the Solution.** We consider the region of the complex velocity $w$ (Fig. 3a), which corresponds to boundary conditions (1). This region representing a circular heptagon with right angles and section $CDC_1$ belongs to the class of polygons in polar (or circular) grids [13–18] which are bounded by the arcs of concentric circles and the segments of straight lines passing through the origin of coordinates.

According to the traditional approach [13, p. 175, and 19], such polygons are transformed, by a logarithmic function, to rectangular ones with the subsequent use of the Christoffel–Schwartz formula. However, this method increases the number of conformal-mapping parameters. This involves difficulties associated with the disturbed conform-

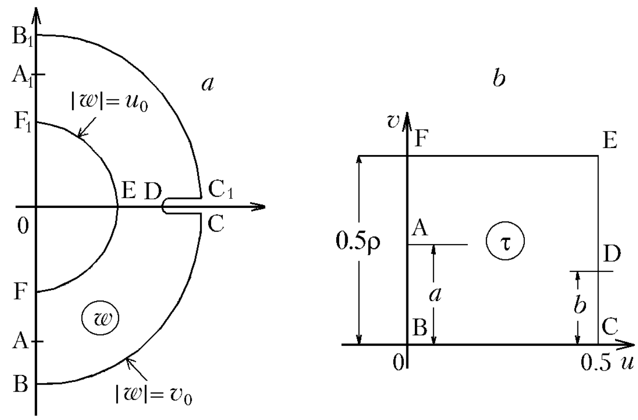

Fig. 3. Region of the complex velocity (a) and the auxiliary parametric variable (b).

ity of mapping at the critical points $w = 0$ and $w = \infty$ of the logarithmic function used. The total number of unknown constants increases even more, when we have to allow for the parameters appearing in solution of the filtration problem itself. We are dealing with the scale modeling constant and the affixes of removable singular points occurring on the $z$ flow plane and absent from the $w$ plane.

Unlike the universally adopted method, below we propose that the Riemann–Schwarz symmetry principal be used. Conformal mapping is realized directly in a closed form simple and convenient for subsequent practical purposes in terms of elementary functions; all the mapping parameters sought are determined incidentally during the construction of the solution.

In view of the total symmetry, we restrict our consideration to the region of motion ABCDEF on the $z$, $\omega$, and $w$ planes (Fig. 1) and the corresponding similar regions on the $\omega$ and $w$ planes (Figs. 2 and 3a).

With allowance for the special properties of the polygons in polar grids due to the abundance of right angles, in conformal mapping, it is convenient to take, as a canonical region, the rectangle [20] $0 < \mathrm{Re}\,\tau < 0.5$, $0 < \mathrm{Im}\,\tau < 0.5\rho$ of the $\tau$ plane (Fig. 3b), where $\rho(k) = K'/K$, $K' = K(k')$; $k' = \sqrt{1-k^2}$; $K(k)$ is the complete elliptic integral of the first kind for the modulus $k$. Then the function conformally mapping this rectangle onto the quarter of the ring of the plane of the complex velocity $w$ is expressed as

$$w(\tau) = v_0 \exp(\tau - 0.5)\pi i\,, \tag{2}$$

whence we determine the physical parameter $u_0 = v_0 \exp(-0.5\pi\rho)$.

We conformally map the rectangle of the auxiliary variable $\tau$ onto the region of the complex potential $\omega$ (Fig. 2). As a result we obtain

$$\omega = \frac{0.5}{K(k)} F\left[\arcsin \frac{\lambda\,\mathrm{dn}\,(2K\tau, k)}{k\sqrt{1-\lambda^2\,\mathrm{sn}^2\,(2K\tau, k)}}, m\right], \tag{3}$$

where $F(\varphi, m)$ is the elliptic integral of the first kind for the modulus $m = k\sqrt{(1-k'^2A^2B^2)/(1-k'^2A^2)\lambda^2}$, $\lambda = \sqrt{(1-k'^2B^2)}$, $A = \mathrm{sn}\,(2Ka, k')$, $B = \mathrm{sn}\,(2Kb, k')$, and $\mathrm{sn}\,(\varphi, k)$, $\mathrm{cn}\,(\varphi, k)$, and $\mathrm{dn}\,(\varphi, k)$ are the Jacobi elliptic functions (sine, cosine, and delta respectively) of the modulus $k$. The filtration flow rate is determined from the formula

$$Q = 0.5H\rho\,(m) = 0.5HK'(m)/K(m)\,. \tag{4}$$

To solve the problem we use the first variant of the velocity-hodograph method [10, pp. 250–251, 11, p. 60, and 12, pp. 603–606]. Taking relations (2) and (3) into account and following [21, 22], we arrive at the dependences

$$\frac{d\omega}{d\tau} = -\frac{C\,\mathrm{sn}\,(2K\tau, k)\,\mathrm{cn}\,(2K\tau, k)}{\Delta(\tau)}\,, \quad \frac{dz}{d\tau} = -\frac{C\,\mathrm{sn}\,(2K\tau, k)\,\mathrm{cn}\,(2K\tau, k)\exp((0.5-\tau)\pi i)}{v_0\Delta(\tau)}\,, \tag{5}$$

$$\Delta(\tau) = \sqrt{[1-\lambda^2 \operatorname{sn}^2(2K\tau, k)]\,[A^2 + (1-A^2)\operatorname{sn}^2(2K\tau, k)]}\ ,$$

where $C > 0$ is the scale modeling constant. Functions (5) satisfy boundary conditions (1) formulated in terms of the functions $d\omega/d\tau$ and $dz/d\tau$; therefore, they are the parametric solution of the initial boundary-value problem. Representation of dependences (5) for different portions of the boundary of the $\tau$ region with subsequent integration over the entire contour of the auxiliary region leads to closure of the contour of the region of the motion $z$, thus serving as a computation check.

As a result we obtain the expressions for the basic geometric and filtration characteristics

$$\frac{C}{v_0}\int_0^{0.5} X_{\mathrm{BC}}dt = \Delta l\ ,\quad \frac{C}{v_0}\int_0^{0.5} Y_{\mathrm{BC}}dt = \Delta d\ ,\quad C\int_0^{0.5} \Phi_{\mathrm{EF}}dt = 0.5H\ , \tag{6}$$

the coordinates of the points of the apron's underground contour BC

$$x_{\mathrm{BC}}(t) = l - \frac{C}{v_0}\int_0^{t} X_{\mathrm{BC}}dt\ ,\quad y_{\mathrm{BC}}(t) = -d_1 - \frac{C}{v_0}\int_0^{t} Y_{\mathrm{BC}}dt\ ,\quad 0 \le t \le 0.5\ , \tag{7}$$

and the coordinates of the curvilinear confining layer EF

$$x_{\mathrm{EF}}(t) = L - \frac{C}{u_0}\int_0^{t} X_{\mathrm{EF}}dt\ ,\quad y_{\mathrm{EF}}(t) = -\frac{C}{u_0}\int_0^{t} Y_{\mathrm{EF}}dt\ ,\quad 0 \le t \le 0.5\ . \tag{8}$$

Here, $\Delta l = l - l_1$, $\Delta d = d - d_1$, $X_{\mathrm{BC}} = \sin \pi t \dfrac{\operatorname{sn}(2Kt, k)\operatorname{cn}(2Kt, k)}{\Delta(t)}$, $Y_{\mathrm{BC}} = \cos \pi t \dfrac{\operatorname{sn}(2Kt, k)\operatorname{cn}(2Kt, k)}{\Delta(t)}$, $\Phi_{\mathrm{EF}} = \dfrac{\operatorname{dn}(2Kt, k')}{\Delta_1(t)}$, $L = l + l_2$, $X_{\mathrm{EF}} = \sin \pi t\Phi_{\mathrm{EF}}$, $Y_{\mathrm{EF}} = \cos \pi t\Phi_{\mathrm{EF}}$, $\Delta_1 = \sqrt{[\operatorname{dn}^2(2Kt, k') - \lambda'^2][1 - A^2\operatorname{dn}^2(2Kt, k')]}$.

Setting $t = 0.5$ in Eqs. (7) and (8), we find the sought dimensions of the underground contour of the apron and the curvilinear confining layer

$$l_1 = x_{\mathrm{BC}}(0.5)\ ,\quad d_1 = y_{\mathrm{BC}}(0.5)\ ,\quad L = l + l_2 = x_{\mathrm{EF}}(0.5)\ ,\quad T = y_{\mathrm{EF}}(0.5)\ . \tag{9}$$

A calculation check uses other expressions for the flow rate $Q$ and the geometric dimensions $l_2$ and $T$

$$Q = C\int_a^{0.5\rho} \Psi_{\mathrm{AF}}dt = C\int_b^{0.5\rho} \Psi_{\mathrm{DE}}dt\ ,\quad l_2 = \frac{C}{v_0}\int_a^{0.5\rho} X_{\mathrm{AF}}dt\ ,\quad T = d + \frac{C}{v_0}\int_b^{0.5\rho} Y_{\mathrm{DE}}dt\ , \tag{10}$$

where

$$\Psi_{\mathrm{AF}} = \frac{\operatorname{sn}(2Kt, k')}{\Delta_2(t)}\ ;\quad \Psi_{\mathrm{DE}} = \frac{\operatorname{sn}(2Kt, k')}{\Delta_3(t)}\ ;\quad X_{\mathrm{AF}} = \Psi_{\mathrm{AF}}\exp \pi t\ ;\quad Y_{\mathrm{DE}} = \Psi_{\mathrm{DE}}\exp \pi t\ ;$$

$$\Delta_2 = \sqrt{[\operatorname{sn}^2(2Kt, k') - A^2]\,[1 - \lambda'^2\operatorname{sn}^2(2Kt, k')]}\ ;\quad \Delta_3 = \sqrt{[\operatorname{sn}^2(2Kt, k') - B^2]\,[1 - k'^2A^2\operatorname{sn}^2(2Kt, k')]}\ .$$

**Limiting Cases. Apron and Rabbet in Flow.** *Flow in the Ground of Infinite Depth.* We primarily dwell on the case where the surface of the confining layer is very deep-seated. Then points E and F merge together at infinity in the plane of motion $z$, and the rectangle of the plane of the auxiliary variable $\tau$ becomes a half-band $0 < \operatorname{Re}\tau < 0.5$, $0 < \operatorname{Lm}\tau < +\infty$ (Fig. 3b), since the modulus $k$ is equal to 0, $k' = 1$, $K(0) = 0.5\pi$, $K'(0) = \infty$, and consequently $\rho = \infty$. But since we have $u_0 = v_0 \exp(-0.5\pi\rho) = 0$, points E and F merge together at the origin of coordinates in the plane of the complex velocity $w$. The solution for this limiting case is obtained from formulas (5)–(10), if we set $k = 0$ in them and take into account that, for this value of the modulus, the elliptic functions degenerate into trigonometric ones: sn $(2K\tau, 0) = \sin \pi\tau$, cn $(2K\tau, 0) = \cos \pi\tau$, and dn $(2K\tau, 0) = 1$. Also we are able to integrate the third equation of (6) and to obtain the modeling parameters in explicit form

$$C = 0.5H\sqrt{(1 - B^2)(1 - A^2)}\,\pi\,. \tag{11}$$

Formulas (9) and (10) yield that the flow rate is $Q = \infty$, $l_2 = \infty$, and $T = \infty$. We note two limiting cases associated with the degeneration of the conformal-mapping parameters A and B within the framework of the filtration diagram in question.

*Apron in Flow (Not Sunk) with a Horizontal Insert.* In this case the vertical segment AB is absent from the flow plane, which corresponds to the merging of points A and B: then we have the parameters $a = A = 0$ and $d_1 = 0$.

Integrating Eqs. (6), we obtain the following expressions for the filtration characteristics:

$$\Delta l = \frac{H(1 - \lambda')}{\pi v_0 \lambda}, \quad \Delta d = d = \frac{H[E(\lambda) - \lambda'^2 K(\lambda)]}{\pi v_0 \lambda}, \quad l_1 = \frac{H[E(\lambda') - \lambda^2 K(\lambda') + \lambda']}{\pi v_0 \lambda},$$
$$l = l_1 + \Delta l = \frac{H[E(\lambda') - \lambda^2 K(\lambda') + 1]}{\pi v_0 \lambda}, \tag{12}$$

where $E(\lambda)$ is the complete elliptic integral of the second order for the modulus $\lambda = \sqrt{1 - B^2}$. Formulas (12) are coincident (accurate to the notation) with formulas (10.9), (10.13), and (10.16) in [5, pp. 197–198].

*Rabbet or Cutoff Apron in Flow.* In this case the horizontal segment CD is absent from the flow plane, which corresponds to the merging of points C and D: we have the parameters $b = B = 0$ and $l_1 = 0$. Integration of expressions (6) leads here to the expressions

$$\Delta l = l = \frac{H[E(\lambda) - \lambda'^2 K(\lambda)]}{\pi v_0 \lambda}, \quad \Delta d = \frac{H(1 - \lambda')}{\pi v_0 \lambda}, \quad d_1 = \frac{H[E(\lambda') - \lambda^2 K(\lambda') + \lambda']}{\pi v_0 \lambda},$$
$$d = d_1 + \Delta d = \frac{H[E(\lambda') - \lambda^2 K(\lambda') + 1]}{\pi v_0 \lambda}, \tag{13}$$

where the modulus is $\lambda = \sqrt{1 - A^2}$. Formulas (13) are coincident with formulas (10.19), (10.22), and (10.24) from [5, pp. 199–200].

A comparison of formulas (12) for the apron with a horizontal insert in flow and formulas (13) for the rabbet in flow shows that they are obtained from one another by replacement of the parameters $A$ and $d$ in them by $B$ and $l$ and conversely.

**Analysis of Numerical Results.** We evaluate the influence of the model's physical parameters $H$, $v_0$, $l$, and $d$ on the shape and dimensions of the underground contour of the dam from consideration of the two extreme limiting cases of the previous section. Figure 4 shows the apron with a horizontal insert in flow, calculated for $H = 10$, $v_0 = 1$, and $l = 12.4$ (basic variant). Figure 5 plots $d$ (curves 1) and $l_1$ (curves 2) as functions $H$, $v_0$, and $l$.

An analysis of the data of the figures leads us to the following conclusions. Growth in the head and decrease in the velocity of flow past the apron and in its width increase the thickness of the apron and diminish the dimension of the insert. In Fig. 5c, it is seen that the shorter the apron, the thicker it must be, with the same flow velocity being preserved.

Noteworthy is the identical qualitative character of the plots of $d$ and $l_1$ as functions of the parameters $v_0$ and $l$: decrease in the flow velocity $v_0$ and in the apron width $l$ leads to growth in the apron thickness $d$ and decrease in the insert width $l_1$. Thus, with a decrease of 5 times in the velocity $\sigma_0$ the apron thickness grows quite significantly — by 2135%, whereas a change of 1.66 times in the width $l$ is accompanied by a proportional growth of 55% in the thickness $d$ and a decrease of 42.7% in $l_1$. Conversely, as the head $H$ grows twofold, the apron thickness $d$ increases by 252%, whereas the width $l$ is diminished only by 10.9%.

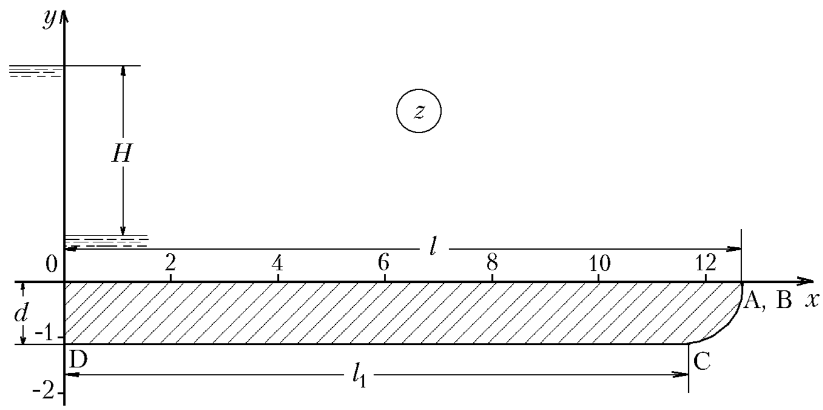


Fig. 4. Apron in flow calculated for $H = 10$, $v_0 = 1$, and $l = 12.4$.

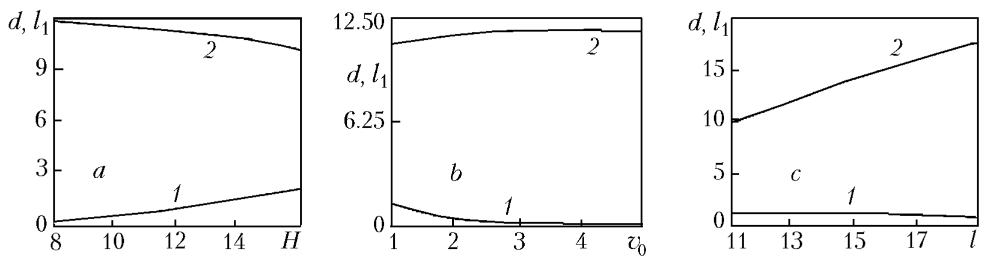


Fig. 5. Quantities $d$ (curves 1) and $l_1$ (curves 2) as functions of $H$ (a), $v_0$ (b), and $l$ (c).

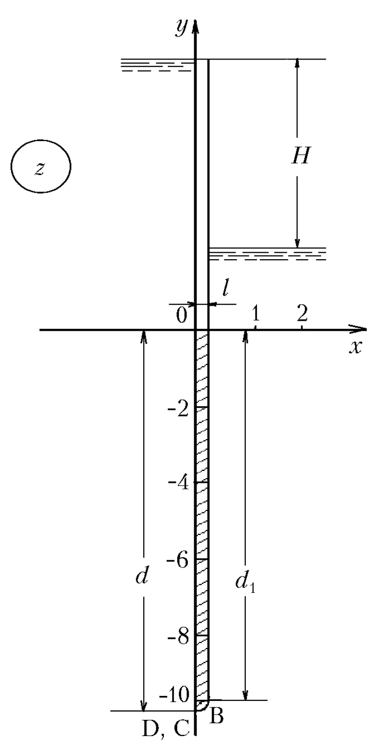


Fig. 6. Rabbet in flow calculated for $H = 10$, $v_0 = 1$, and $d = 10$.

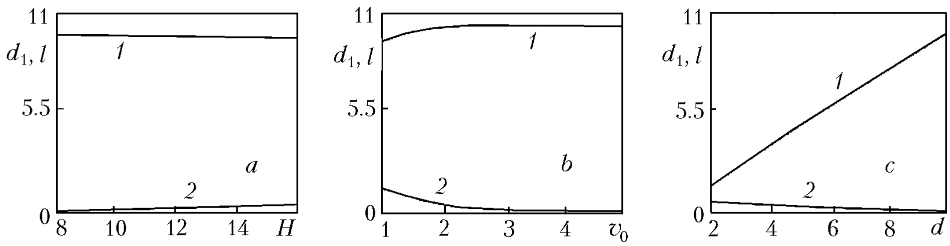


Fig. 7. Quantities $d_1$ (curves 1) and $l$ (curves 2) as functions of $H$ (a), $v_0$ (b), and $d$ (c).

Figure 6 shows the apron in flow calculated for $H$ = 10, $v_0$ = 1, and $d$ = 10 (basic variant). Figure 7 plots the quantities $d_1$ (curves 1) and $l$ (curves 2) as functions of the parameters $H$, $v_0$, and $d$.

An analysis of Fig. 7 shows that the character of the plot of $d_1$ and $l$ as functions of the parameters $H$, $v_0$, and $d$ changes fundamentally here compared to the apron in flow. Now, conversely, growth in the head and decrease in the velocity of flow past the rabbet and in its thickness diminishes the depth and increases the width of the cutoff apron. Figure 7c, where the parameter $d$ is varied, yields that the longer the rabbet, the thinner it must be, with the same flow velocity being preserved. Thus, with an increase of 5 times in the length $d$, the depth $d_1$ grows quite significantly — by 544%, whereas the cutoff-apron width is diminished nearly 4 times.

Just as in the case of the apron, the flow velocity exerts the most substantial influence on the dimensions of the dam. It is seen that the rabbet width grows by 2058% with a decrease of 5 times in $v_0$.

**Conclusions.** We have found the exact analytical solution of the problem on construction of the underground contour of a sunk rectangular apron of water-development works, whose angles are rounded off by the curves of constant filtration rate, in the case where the water-permeable base is underlain by a confining layer with a curvilinear roof, characterized by a constant filtration rate, too. We have studied the limiting cases in detail: the apron with a horizontal insert and the rabbet (cutoff apron) in flow.

It has been established that growth in the head and decrease in the velocity of flow past the apron (rabbet) and in its width (thickness) increase the apron thickness and diminishes the dimension of the insert and, conversely. diminish the depth of the rabbet and increase its width.

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## NOTATION

$a$ and $b$, unknown affixes (images) of points A and D on the plane of the auxiliary parametric variable; $d$, occurrence depth of the apron; $d_1$, depth of the rectangular part of the apron or the rabbet; $E$, complete elliptic integral of the second kind for the modulus $k$; $F$, elliptic integrals of the first kind for the module $k$; $i$, imaginary unit; Im, imaginary part of the complex number; $H$, acting head; $k$, modulus of elliptic integrals; $k'$, additional modulus; $K$ and $K'$, complete elliptic integral of the first kind for the modulus $k$ and $k'$ respectively; $l$, half-width of the apron; $l_1$, half-width of the horizontal underwater part of the dam; $l_2$, width of the water-permeable portion of exit of water into the lower pool; $L$, distance from the origin of coordinates of point F of the confining layer; $m$, modulus of elliptic integrals; $Q$, filtration flow rate of water; Re, real part of the complex number; $t$, integration variable; $T$, largest occurrence depth of the curvilinear confining layer; $u$ and $v$, real and imaginary parts of the auxiliary parametric variable; $u_0$ and $v_0$, filtration rates along the confining layer and the water-development works respectively; $w$, complex flow velocity; $x$, $y$, abscissa and ordinate of a point of the flow region; $z$, complex coordinate of a point of the flow region; $\kappa$, filtration coefficient; $\lambda$ and $\lambda'$, modulus and additional modulus of elliptic integrals respectively; $\rho$, dimensionless quantity related to the ratio of the complete elliptic integrals of the first kind $K'$ and $K$; $\tau$, auxiliary parametric variable; $\varphi$, velocity potential; $\psi$, stream function; $\omega$, complex flow potential. Subscript: 0, fixed constant.

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